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Nilpotent symmetries for a spinning relativistic particle in augmented superfield formalism

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Abstract. The local, covariant, continuous, anticommuting and nilpotent Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetry transformations for *all* the fields of a $(0+1)$-dimensional spinning relativistic particle are obtained in the framework of the augmented superfield approach to the BRST formalism. The trajectory of this super particle, parametrized by a monotonically increasing evolution parameter τ, is embedded in a D-dimensional flat Minkowski spacetime manifold. This physically useful 1-dimensional system is considered on a three $(1+2)$-dimensional supermanifold which is parametrized by an even element (τ) and a couple of odd elements (θ and $\bar{\theta}$) of the Grassmann algebra. Two anticommuting sets of (anti-) BRST symmetry transformations, corresponding to the underlying (super) gauge symmetries for the above system, are derived in the framework of augmented superfield formulation where the horizontality condition, and the invariance of conserved quantities on the (super) manifolds play decisive roles. Geometrical interpretations for the above nilpotent symmetries (and their generators) are provided.

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1 Introduction

For the covariant canonical quantization of gauge theories[1], one of the most elegant and intuitive approaches is the Becchi–Rouet–Stora–Tyutin (BRST) formalism [1,2]. In this formalism, the unitarity and "quantum" gauge (i.e. BRST) invariance are very naturally respected at any arbitrary order of perturbative computations for any arbitrary physical process allowed by the interacting gauge theories (where there exists self-interaction as well as the coupling between the (non-) Abelian gauge field and the matter fields). In fact, the whole strength of the BRST formalism appears in its full blaze of glory in the context of an interacting non-Abelian gauge theory where the (anti-) ghost fields are required in the precise proof of unitarity. To be more accurate, for every gluon loop (Feynman) diagram, one requires a ghost loop diagram so that unitarity of the theory could be maintained at any given order of perturbative calculation (see, e.g., [3] for details). In a modern context, the BRST formalism is indispensable in the realm of topological field theories [4–6], topological string theories [7], string field theories [8], etc. There are well-known connections of this formalism with the mathematics of differential geometry and supersymmetries.

In our present endeavour, we shall be concentrating on the geometrical aspects of the relationship between the BRST formalism and the superfield formalism. To be more elaborate on this topic, it should be noted that, in the framework of the usual superfield formulation [9–14] of the BRST approach to D-dimensional p-form (with $p = 1, 2, \ldots$) Abelian gauge theories, the gauge theory is considered first on a $(D + 2)$-dimensional supermanifold parametrized by the D-number of even (commuting) spacetime x_μ variables (with $\mu = 0, 1, 2, \ldots, D-1$) and a couple of odd (anticommuting) Grassmannian variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). Then, the $(p+1)$-form super curvature $\tilde{F}^{(p+1)} = \tilde{d}\tilde{A}^{(p)}$ is constructed from the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and the super p-form connection $\tilde{A}^{(p)}$ defined on the $(D + 2)$-dimensional supermanifold. This is subsequently equated, due to the so-called horizontality condition [9–14], with the ordinary curvature $(p+1)$-form $F^{(p+1)} = dA^{(p)}$ defined on the D-dimensional ordinary spacetime manifold with the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) and the ordinary p-form connection $A^{(p)}$.

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[1] These theories are endowed with the first-class constraints in the language of Dirac's prescription for the classification of constraints. The local (non-) Abelian 1-form interacting gauge theories provide an almost exact theoretical basis for the three (out of four) fundamental interactions of nature.

The above horizontality condition[2] is christened as the soul-flatness condition in [15] which amounts to setting equal to zero all the Grassmannian components of the (anti-) symmetric super curvature tensor that defines the $(p+1)$-form super curvature.

The process of reduction of the $(p+1)$-form super curvature to the ordinary $(p+1)$-form curvature due to the horizontality condition
(i) generates the nilpotent and anticommuting (anti-) BRST transformations for *only* the gauge fields and the (anti-) ghost fields,
(ii) provides the geometrical interpretation for the nilpotent (anti-) BRST charges as the translational generators $(\text{Lim}_{\bar\theta\to 0}(\partial/\partial\theta))\,\text{Lim}_{\theta\to 0}(\partial/\partial\bar\theta)$ along the $(\theta)\bar\theta$-directions of the $(D+2)$-dimensional supermanifold,
(iii) leads to the geometrical interpretation for the nilpotency property as the two successive translations (i.e. $(\partial/\partial\theta)^2 = (\partial/\partial\bar\theta)^2 = 0$) along either of the Grassmannian directions, and
(iv) captures the anticommutativity of the nilpotent (anti-) BRST charges (and the transformations they generate) in the relationship $(\partial/\partial\theta)(\partial/\partial\bar\theta) + (\partial/\partial\bar\theta)(\partial/\partial\theta) = 0$. It should be re-emphasized, however, that all these nice geometrical connections between the BRST formalism and the usual superfield formalism [9–14] are confined *only* to the gauge fields and the (anti-) ghost fields of a BRST invariant Lagrangian density of the D-dimensional interacting p-form Abelian gauge theory. The matter fields of an *interacting* D-dimensional p-form Abelian gauge theory remain untouched in the above superfield formalism as far as their nilpotent and anticommuting (anti-) BRST symmetry transformations are concerned.

The above constraint due to the horizontality condition (where only $\tilde d$ and d play important roles) has been generalized to the constraints that emerge from the full use of super $(\tilde d, \tilde\delta, \tilde\Delta)$ and ordinary (d, δ, Δ) de Rham cohomological operators (see, e.g., [16–20] for details). This complete set of restrictions on the (super) manifolds leads to the existence of (anti-) BRST, (anti-) co-BRST and a bosonic (which is equal to the anticommutator of the (anti-) BRST and the (anti-) co-BRST) symmetry transformations *together* for the $(1+1)$-dimensional non-interacting 1-form (non-) Abelian gauge theories. In the Lagrangian formulation, the above kind of symmetries have also been shown to exist for the $(3+1)$-dimensional free Abelian 2-form gauge theory [21, 22]. There exists a discrete symmetry transformation for the above field theoretical models (in the Lagrangian formulation) which corresponds to the Hodge duality $*$ operation of differential geometry. Thus, the above models do provide a tractable set of field theoretical examples for the Hodge theory. It is worthwhile to pinpoint, however, that even the above new attempts of the superfield formalism (with the full set of cohomological operators) do *not* shed any light on the nilpotent symmetry transformations associated with the matter fields.

In a set of recent papers [23–27], the above usual superfield formalism (with the theoretical arsenal of horizontality condition and its generalizations) has been augmented to include the invariance of the conserved currents and/or charges on the (super) manifolds. The latter constraints, on the (super) manifolds, lead to the derivation of the nilpotent (anti-) BRST symmetry transformations for the matter fields of the interacting 4-dimensional 1-form (non-) Abelian gauge theories. We christen this extended version of the superfield formalism as *augmented* superfield formulation applied to the 4-dimensional interacting 1-form gauge theory described by the (anti-) BRST invariant Lagrangian density. It is worth emphasizing that, in the framework of augmented superfield formalism, all the geometrical interpretations, listed in the previous paragraph, remain intact. As a consequence, there is a very nice mutual complementarity between the old constraint (i.e. the horizontality condition) and new constraint(s) on the (super) manifolds. We do obtain, as a bonus and by-product, all the nilpotent (anti-) BRST transformations for *all* the fields (i.e. gauge fields, (anti-) ghost fields and matter fields) of an interacting 1-form gauge theory.

The purpose of the present paper is to derive the nilpotent (anti-) BRST transformations for *all* the fields, present in the description of a free spinning relativistic particle (moving on a super world-line) in the framework of the augmented superfield formulation [23–27]. Our present endeavour is essential primarily on four counts:

First and foremost, this formalism is being applied to a supersymmetric system for the first time. It is worth pointing out that its non-supersymmetric counterpart (i.e. the system of a free scalar relativistic particle) has already been discussed in the framework of the augmented superfield formulation in our earlier work [27].

Second, to check the mutual consistency and complementarity between
(i) the horizontality condition, and
(ii) the invariance of conserved quantities on the (super) manifolds for this physical system. These were found to be true in the cases of
(a) a free scalar relativistic particle [27],
(b) the interacting (non-) Abelian gauge theories in two $(1+1)$-dimensions (2D) [23, 24],
(iii) the interacting (non-) Abelian gauge theories in four $(3+1)$-dimensions (4D) of spacetime [25, 26].

Third, to generalize our earlier works [23–27], which were connected *only* with the gauge symmetries and reparametrization symmetries, to the case where the supergauge symmetry also exists for the present system under discussion.

Finally, to tap the potential and power of the above restrictions in the derivation of the nilpotent symmetries for the case of a new system where the fermionic as well as bosonic
(i) gauge fields (i.e. χ, e), and

[2] For the 1-form non-Abelian gauge theory, the horizontality condition $\tilde F^{(2)} = F^{(2)}$, where 2-form super curvature $\tilde F^{(2)} = \tilde d\tilde A^{(1)} + \tilde A^{(1)}\wedge\tilde A^{(1)}$ and 2-form ordinary curvature $F^{(2)} = dA^{(1)} + A^{(1)}\wedge A^{(1)}$, leads to the exact derivation of the nilpotent and anticommuting (anti-) BRST symmetry transformations for the non-Abelian gauge field and the corresponding (anti-) ghost fields of the theory (see, e.g. [12] for details).

(ii) (anti-) ghost fields (i.e. $(\bar{c})c)$ and $(\bar{\beta})\beta$) do exist in the Lagrangian description of this *supersymmetric* system (cf. (2.7) below).

The contents of our present paper are organized as follows. In Sect. 2, we very clearly discuss the essentials of the reparametrization, gauge and supergauge symmetry transformations for the spinning massive relativistic particle in the Lagrangian formulation. Two sets of anticommuting BRST symmetry transformations, that exist for the above system under a very specific limit, are also discussed in this section. Sections 3 and 4 are the central parts of our paper. Section 3 is devoted to the derivation of the nilpotent (anti-) BRST symmetry transformations (corresponding to the gauge symmetry transformations) in the framework of the augmented superfield formalism. In the forthcoming section (i.e. Sect. 4), for the first time, we extend the idea of the augmented superfield formalism to obtain the (anti-) BRST symmetry transformations (corresponding to the supergauge symmetry transformations) that exist for the spinning relativistic particle. Finally, in Sect. 5, we make some concluding remarks and point out a few future directions for further investigations.

2 Preliminary: nilpotent BRST symmetries

Let us begin with the various equivalent forms of the reparametrization invariant Lagrangians for the description of a free massive spinning relativistic particle moving on a super world-line that is embedded in a D-dimensional flat Minkowski target spacetime manifold. These, triplets of appropriate Lagrangians, are [28, 29]:

$$
\begin{aligned}
L_0^{(m)} &= m[(\dot{x}_\mu + \mathrm{i}\chi\psi_\mu)^2]^{1/2} + \frac{\mathrm{i}}{2}\left(\psi_\mu\dot{\psi}^\mu - \psi_5\dot{\psi}_5\right) \\
&\quad -\mathrm{i}\chi\psi_5 m, \\
L_\mathrm{f}^{(m)} &= p_\mu\dot{x}^\mu - \frac{1}{2}e\left(p^2 - m^2\right) + \frac{\mathrm{i}}{2}\left(\psi_\mu\dot{\psi}^\mu - \psi_5\dot{\psi}_5\right) \\
&\quad +\mathrm{i}\chi\left(\psi_\mu p^\mu - \psi_5 m\right), \\
L_\mathrm{s}^{(m)} &= \frac{1}{2}e^{-1}\left(\dot{x}_\mu + \mathrm{i}\chi\psi_\mu\right)^2 + \frac{1}{2}em^2 \\
&\quad +\frac{\mathrm{i}}{2}\left(\psi_\mu\dot{\psi}^\mu - \psi_5\dot{\psi}_5\right) - \mathrm{i}\chi\psi_5 m. \qquad (2.1)
\end{aligned}
$$

In the above, the mass-shell condition $p^2 - m^2 = 0$, the constraint condition $p \cdot \psi - m\psi_5 = 0$ and the force free (i.e. $\dot{p}_\mu = 0$) motion of the *spinning* relativistic particle are some of the key common features for

(i) the Lagrangian with the square root $L_0^{(m)}$,

(ii) the first-order Lagrangian $L_\mathrm{f}^{(m)}$, and

(iii) the second-order Lagrangian $L_\mathrm{s}^{(m)}$. The constraints $p^2 - m^2 \approx 0$ and $p \cdot \psi - m\psi_5 \approx 0$ in $L_\mathrm{f}^{(m)}$ are taken care of by the Lagrange multiplier fields $e(\tau)$ and $\chi(\tau)$ (with $\chi^2 = 0$) which are

(a) the bosonic and fermionic gauge fields of the present system, respectively, and

(b) the analogues of the vierbein and Rarita–Schwinger (gravitino) fields in the language of the supergravity theories. The Lorentz vector fermionic fields $\psi_\mu(\tau)$ (with $\mu = 0, 1, 2, \ldots, D-1$) are the superpartner of the target space coordinate variable $x_\mu(\tau)$ (with $\mu = 0, 1, 2, \ldots, D-1$) and classically they present spin degrees of freedom. Furthermore, they anticommute with themselves (i.e. $\psi_\mu\psi_\nu + \psi_\nu\psi_\mu = 0$) and other fermionic field variables (i.e. $\psi_\mu\psi_5 + \psi_5\psi_\mu = 0, \psi_\mu\chi + \chi\psi_\mu = 0$) of the system under consideration. The τ-independent mass parameter m (i.e. the analogue of the cosmological constant term) is introduced in our present system through the anticommuting (i.e. $\psi_5\chi + \chi\psi_5 = 0, (\psi_5)^2 = -1$, etc.) Lorentz scalar field $\psi_5(\tau)$. The momenta $p_\mu(\tau)$ (with $\mu = 0, 1, 2, \ldots, D-1$), present in $L_\mathrm{f}^{(m)}$, are canonically conjugate to the target space coordinate variable $x^\mu(\tau)$. It is evident that, except for the mass parameter m, the rest of the field variables are the functions of monotonically increasing parameter τ that characterizes the trajectory (i.e. the super world-line) of the massive spinning relativistic particle. Here $\dot{x}^\mu = (dx^\mu/d\tau) = ep^\mu - \mathrm{i}\chi\psi^\mu$, $\dot{\psi}_\mu = (d\psi_\mu/d\tau) = \chi p_\mu, \dot{\psi}_5 = (d\psi_5/d\tau) = \chi m$ are the generalized versions of "velocities" of the massive spinning relativistic particle.

In what follows, we shall focus on the first-order Lagrangian $L_\mathrm{f}^{(m)}$ for the discussion of the symmetry properties of the system. This is due to the fact that this Lagrangian is comparatively simpler in the sense that there are no square roots and there are no field variables in the denominator. Furthermore, it is endowed with the maximum number of field variables and, therefore, is interesting from the point of view of theoretical discussions. Under an infinitesimal version of the reparametrization transformations $\tau \to \tau' = \tau - \epsilon(\tau)$, where $\epsilon(\tau)$ is an infinitesimal parameter, the field variables of $L_\mathrm{f}^{(m)}$ transform as

$$
\begin{aligned}
&\delta_r x_\mu = \epsilon\dot{x}_\mu, \quad \delta_r p_\mu = \epsilon\dot{p}_\mu, \qquad \delta_r\psi_\mu = \epsilon\dot{\psi}_\mu, \qquad (2.2)\\
&\delta_r\psi_5 = \epsilon\dot{\psi}_5, \quad \delta_r\chi = \frac{d}{d\tau}(\epsilon\chi), \quad \delta_r e = \frac{d}{d\tau}(\epsilon e).
\end{aligned}
$$

It should be noted that

(i) $\delta_r\Sigma(\tau) = \Sigma'(\tau) - \Sigma(\tau)$ for the generic field variable $\Sigma = x_\mu, p_\mu, e, \psi_\mu, \psi_5, \chi$, and

(ii) the gauge fields e and χ do transform in a similar fashion (and distinctly different from the rest of the field variables). The first- and the second-order Lagrangians are endowed with the first-class constraints $\Pi_e \approx 0, \Pi_\chi \approx 0, p^2 - m^2 \approx 0, p \cdot \psi - m\psi_5 \approx 0$ in the language of Dirac's prescription for the classification of constraints. Here Π_e and Π_χ are the canonical conjugate momenta corresponding to the einbein field $e(\tau)$ and the fermionic gauge field $\chi(\tau)$, respectively. There are second-class constraints too in the theory but we shall *not* concentrate on them for our present discussion. The existence of the first-class constraints $\Pi_e \approx 0$ and $p^2 - m^2 \approx 0$ on this physical system, generates the following gauge symmetry transformation δ_g for the field variables of the first-order Lagrangian $L_\mathrm{f}^{(m)}$ (for the description of a spinning relativistic particle):

$$\delta_g x_\mu = \xi p_\mu, \quad \delta_g p_\mu = 0, \quad \delta_g \psi_\mu = 0, \quad \delta_g \psi_5 = 0, \quad \delta_g \chi = 0, \quad \delta_g e = \dot{\xi}, \tag{2.3}$$

where $\xi(\tau)$ is an infinitesimal gauge parameter. The pair of fermionic constraints $\pi_\chi \approx 0$ and $p \cdot \psi - m\psi_5 \approx 0$ generate the following supergauge symmetry transformations δ_{sg} for the bosonic and fermionic field variables of the first-order Lagrangian $L_{\rm f}^{(m)}$:

$$\delta_{sg} x_\mu = \kappa \psi_\mu, \quad \delta_{sg} p_\mu = 0, \quad \delta_{sg} \psi_\mu = \mathrm{i}\kappa p_\mu, \quad \delta_{sg} \psi_5 = \mathrm{i}\kappa m, \quad \delta_{sg} \chi = \mathrm{i}\dot{\kappa}, \quad \delta_{sg} e = 2\kappa\chi, \tag{2.4}$$

where $\kappa(\tau)$ is an infinitesimal fermionic (i.e. $\kappa^2 = 0$) supergauge transformation parameter. The above infinitesimal transformations are *symmetry* transformations because

$$\delta_r L_{\rm f}^{(m)} = \frac{d}{d\tau}\left[\epsilon L_{\rm f}^{(m)}\right], \quad \delta_g L_{\rm f}^{(m)} = \frac{d}{d\tau}\left[\frac{\xi}{2}\left(p^2 + m^2\right)\right],$$
$$\delta_{sg} L_{\rm f}^{(m)} = \frac{d}{d\tau}\left[\frac{\kappa}{2}\left(p \cdot \psi + m\psi_5\right)\right]. \tag{2.5}$$

It is straightforward to check that, for the generic field variable Σ, we have $(\delta_g - \mathrm{i}\delta_{sg})\Sigma = \delta_r \Sigma$ with the identifications $\xi = e\epsilon$ and $\kappa = \chi\epsilon$ and validity of the on-shell conditions (i.e. $\dot{p}_\mu = 0, \dot{\psi}_\mu = \chi p_\mu, \dot{\psi}_5 = \chi m, \dot{x}_\mu = e p_\mu - \mathrm{i}\chi\psi_\mu, p^2 = m^2, p \cdot \psi = m\psi_5$).

The gauge and supergauge symmetry transformations (2.3) and (2.4) can be combined together and generalized to the nilpotent BRST symmetry transformations. The usual trick of the BRST prescription could be exploited here to express the gauge and supergauge parameters $\xi = \eta c$ and $\kappa = \eta\beta$ in terms of the fermionic ($c^2 = 0$) and bosonic ($\beta^2 \neq 0$) ghost fields and η. It will be noted that η is the spacetime independent anticommuting (i.e. $\eta c + c\eta = 0$, etc.) parameter which is required to maintain the bosonic nature of ξ (in $\xi = \eta\, c$) and the fermionic nature of κ (in $\kappa = \eta\beta$). The ensuing nilpotent ($(s_b^{(0)})^2 = 0$) BRST transformations [29], for the spinning relativistic particle, are[3]

$$\begin{aligned}
&s_b^{(0)} x_\mu = c p_\mu + \beta\psi_\mu, \quad s_b^{(0)} c = -\mathrm{i}\beta^2, \quad s_b^{(0)} p_\mu = 0,\\
&s_b^{(0)} \psi_\mu = \mathrm{i}\beta p_\mu, \quad s_b^{(0)} \bar{c} = \mathrm{i}b, \quad s_b^{(0)} b = 0,\\
&s_b^{(0)} e = \dot{c} + 2\beta\chi, \quad s_b^{(0)} \chi = \mathrm{i}\dot{\beta}, \quad s_b^{(0)} \psi_5 = \mathrm{i}\beta m,\\
&s_b^{(0)} \beta = 0, \quad s_b^{(0)} \bar{\beta} = \mathrm{i}\gamma, \quad s_b^{(0)} \gamma = 0.
\end{aligned} \tag{2.6}$$

[3] We follow here the notation and conventions adopted in [30, 31]. In its full blaze of glory, the true nilpotent (anti-) BRST transformations $\delta_{(A)B}$ are the product of an (anticommuting) spacetime-independent parameter η and the nilpotent transformations $s_{(a)b}$. It is clear that η commutes with all the bosonic (even) fields of the theory and anticommutes with fermionic (odd) fields (i.e. $\eta c + c\eta = 0, \eta\bar{c} + \bar{c}\eta = 0$, etc.).

The above off-shell nilpotent ($(s_b^{(0)})^2 = 0$) transformations are the *symmetry* transformations for the system because the Lagrangian (which is the generalization of $L_{\rm f}^{(m)}$)

$$\begin{aligned}
L_b^{(m)} &= p \cdot \dot{x} - \frac{1}{2} e\left(p^2 - m^2\right) + \frac{\mathrm{i}}{2}\left(\psi \cdot \dot{\psi} - \psi_5 \dot{\psi}_5\right)\\
&\quad + \mathrm{i}\chi\left(p \cdot \psi - \psi_5 m\right)\\
&\quad + b\dot{e} + \gamma\dot{\chi} + \frac{1}{2} b^2 - \mathrm{i}\dot{\bar{c}}\left(\dot{c} + 2\chi\beta\right) - \dot{\bar{\beta}}\dot{\beta}
\end{aligned} \tag{2.7}$$

transforms to a total derivative under (2.6), namely

$$s_b^{(0)} L_b^{(m)} = \frac{d}{d\tau}\Big[\frac{1}{2} c\left(p^2 + m^2\right) + \frac{1}{2}\beta(p \cdot \psi + m\psi_5) + b(\dot{c} + 2\beta\chi) - \mathrm{i}\gamma\dot{\beta}\Big]. \tag{2.8}$$

A few comments are in order now. First, the first-order Lagrangian $L_{\rm f}^{(m)}$ has been extended to include the gauge-fixing term and the Faddeev–Popov ghost terms (constructed by the bosonic as well as the fermionic (anti-) ghost fields) in $L_b^{(m)}$. Second, the bosonic auxiliary field b and the fermionic auxiliary field γ (with $\gamma^2 = 0, \gamma\chi + \chi\gamma = 0, c\gamma + \gamma c = 0$, etc.) are the Nakanishi–Lautrup fields. Third, the fermionic (i.e. $\bar{c}^2 = 0, c\bar{c} + \bar{c}c = 0$, etc.) anti-ghost field $\bar{c}$ and the bosonic (i.e. $\bar{\beta}^2 \neq 0, \beta\bar{\beta} = \bar{\beta}\beta$, etc.) anti-ghost field $\bar{\beta}$ are required in the theory to have a precise nilpotent BRST symmetry for the system under consideration. Fourth, the above nilpotent transformations (2.6) are generated by the conserved ($\dot{Q}_b^{(0)} = 0$) and nilpotent (i.e. $\left(Q_b^{(0)}\right)^2 = 0$) BRST charge $Q_b^{(0)}$ as given below:

$$Q_b^{(0)} = \frac{c}{2}\left(p^2 - m^2\right) + \beta\left(p \cdot \psi - m\psi_5\right) + b\left(\dot{c} + 2\beta\chi\right) + \dot{\bar{c}}\beta^2 - \mathrm{i}\gamma\dot{\beta}. \tag{2.9}$$

Fifth, the nilpotent ($(s_{ab}^{(0)})^2 = 0$) anti-BRST symmetry transformations $s_{ab}^{(0)}$ and corresponding generator $Q_{ab}^{(0)}$ can be computed from (2.6) and (2.9) by the substitutions $c \leftrightarrow \bar{c}$ and $\beta \leftrightarrow \bar{\beta}$. Sixth, the above generators and corresponding symmetries obey the property of anticommutativity (i.e. $s_b^{(0)} s_{ab}^{(0)} + s_{ab}^{(0)} s_b^{(0)} = 0, Q_b^{(0)} Q_{ab}^{(0)} + Q_{ab}^{(0)} Q_b^{(0)} = 0$). Finally, the conservation of the BRST charge $Q_b^{(0)}$ can be proven by exploiting the equations of motion

$$\begin{aligned}
&\dot{p}_\mu = 0, \quad \dot{x}_\mu = e p_\mu - \mathrm{i}\chi\psi_\mu, \quad \dot{\psi}_\mu = \chi p_\mu, \quad \dot{\psi}_5 = \chi m,\\
&\dot{\chi} = 0, \quad b = -\dot{e}, \quad \dot{b} = -\frac{1}{2}\left(p^2 - m^2\right), \quad \ddot{\beta} = 0,\\
&\ddot{\bar{\beta}} = 2\mathrm{i}\dot{\bar{c}}\chi, \quad \ddot{\bar{c}} = 0, \quad \ddot{c} + 2\dot{\beta}\chi + 2\beta\dot{\chi} = 0,\\
&\dot{\gamma} + 2\mathrm{i}\dot{\bar{c}}\beta + \mathrm{i}(p \cdot \psi - m\psi_5) = 0,
\end{aligned} \tag{2.10}$$

derived from the BRST invariant Lagrangian $L_b^{(m)}$ of (2.7).

For our further discussion, we deal with the limiting cases of (2.6) and (2.7) so that we can study the BRST transformations corresponding to the gauge transformations (2.3) and the supergauge transformations (2.4), separately and independently. It is evident that $\beta \to 0, \bar{\beta} \to 0, \gamma \to 0$ in (2.6) leads to the *nilpotent* $((s_b^{(1)})^2 = 0)$ BRST transformations $s_b^{(1)}$, corresponding to the gauge transformations (2.3):

$$s_b^{(1)} x_\mu = c p_\mu, \quad s_b^{(1)} p_\mu = 0, \quad s_b^{(1)} c = 0, \quad s_b^{(1)} \psi_\mu = 0,$$
$$s_b^{(1)} \psi_5 = 0, \quad s_b^{(1)} \bar{c} = \mathrm{i} b, \quad s_b^{(1)} b = 0, \quad s_b^{(1)} \chi = 0,$$
$$s_b^{(1)} e = \dot{c}, \tag{2.11}$$

which are the symmetry transformations for the Lagrangian

$$L_b^{(1)} = p \cdot \dot{x} - \frac{1}{2} e \left(p^2 - m^2\right) + \frac{\mathrm{i}}{2} \left(\psi \cdot \dot{\psi} - \psi_5 \dot{\psi}_5\right)$$
$$+ \mathrm{i}\chi \left(p \cdot \psi - \psi_5 m\right) + b\dot{e} + \frac{1}{2} b^2 - \mathrm{i}\dot{\bar{c}}\dot{c}, \tag{2.12}$$

obtained from (2.7) under the above specific limits (i.e. $(\beta, \bar{\beta}, \gamma) \to 0$). In another limiting case (i.e. $c \to 0, \bar{c} \to 0, b \to 0$) of (2.6), we obtain the following *non-nilpotent* $((s_b^{(2)})^2 \neq 0)$ BRST transformations, corresponding to the supergauge transformations (2.4):

$$s_b^{(2)} x_\mu = \beta \psi_\mu, \quad s_b^{(2)} p_\mu = 0, \quad s_b^{(2)} \beta = 0,$$
$$s_b^{(2)} \psi_\mu = \mathrm{i}\beta p_\mu, \quad s_b^{(2)} \psi_5 = \mathrm{i}\beta m, \quad s_b^{(2)} \chi = \mathrm{i}\dot{\beta},$$
$$s_b^{(2)} \bar{\beta} = \mathrm{i}\gamma, \quad s_b^{(2)} \gamma = 0, \quad s_b^{(2)} e = 2\beta\chi, \tag{2.13}$$

that are found to be the symmetry transformations for the Lagrangian

$$L_b^{(2)} = p \cdot \dot{x} - \frac{1}{2} e \left(p^2 - m^2\right) + \frac{\mathrm{i}}{2} \left(\psi \cdot \dot{\psi} - \psi_5 \dot{\psi}_5\right)$$
$$+ \mathrm{i}\chi \left(p \cdot \psi - \psi_5 m\right) + \gamma \dot{\chi} - \dot{\bar{\beta}}\dot{\beta}, \tag{2.14}$$

derived from (2.7) under the above limits: $(c, \bar{c}, b) \to 0$. It will be noted that
(i) the anti-BRST versions of (2.11) and (2.13) can be obtained by the substitutions $c \leftrightarrow \bar{c}$ and $\beta \leftrightarrow \bar{\beta}$, respectively;
(ii) in a similar fashion, the generators $Q_b^{(1,2)}$ for $s_b^{(1,2)}$, can be derived from (2.9) by taking into account the above limiting cases;
(iii) the equations of motion for the Lagrangians (2.12) and (2.14) can be derived from (2.10) under the limits cited above.

We wrap up this section with a couple of comments on the *nilpotency* property of the BRST transformations $s_b^{(2)}$ in (2.13) that correspond to the supergauge transformations in (2.4). First, it is straightforward to note that, under the restriction $\beta^2 = 0$, one can restore the nilpotency (i.e. $(s_b^{(2)})^2 = 0$) for the transformations $s_b^{(2)}$. This aspect of β can be fulfilled if this bosonic ghost field β is taken to be a composite (i.e. $\beta \sim c_1 c_2$) of a couple of fermionic $(c_1^2 = c_2^2 = 0, c_1 c_2 + c_2 c_1 = 0)$ ghost fields c_1 and c_2 [29]. Second, if this condition (i.e. $\beta^2 = 0$) is true, it turns out that the two BRST symmetry transformations $s_b^{(1,2)}$ really decouple from each other in the sense that $\{s_b^{(1)}, s_b^{(2)}\} = 0$. In such a situation, they can distinctly be separate from each other as well as from s_b in (2.6) (with $(s_b^{(1,2)})^2 = 0, \{s_b^{(1)}, s_b^{(2)}\} = 0$) [29]. It is, ultimately, interesting to note that *all* the nilpotent symmetry transformations $s_r^{(i)}$ (with $i = 0, 1, 2$), for the generic field Σ of the system, can be succinctly expressed in terms of the conserved and nilpotent charges $Q_r^{(i)}$ (with $i = 0, 1, 2$) as

$$s_r^{(i)} \Sigma = -\mathrm{i}[\Sigma, Q_r^{(i)}]_\pm, \tag{2.15}$$
$$r = b, ab, \quad \Sigma = x_\mu, p_\mu, b, e, c, \bar{c}, \chi, \gamma, \beta, \bar{\beta}, \psi_\mu, \psi_5.$$

The $\pm$ signs, present as the subscripts on the above square brackets, stand for the brackets to be (anti-) commutators for the generic field Σ being (fermionic) bosonic in nature.

3 Gauge BRST symmetries: augmented superfield approach

To derive the nilpotent $((s_{(a)b}^{(1)})^2 = 0)$ (anti-) BRST transformations $s_{(a)b}^{(1)}$ (cf. (2.11)) for the gauge (einbein) field $e(\tau)$ and the (anti-) ghost fields $(\bar{c})c$ in the superfield formalism (where the horizontality condition plays a decisive role), we begin with a general three $(1+2)$-dimensional supermanifold parametrized by the superspace coordinates $Z = (\tau, \theta, \bar{\theta})$ where τ is an even (bosonic) coordinate and θ and $\bar{\theta}$ are the two odd (Grassmannian) coordinates (with $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). On this supermanifold, one can define a 1-form supervector superfield $\tilde{V} = dZ(\tilde{A})$ with $\tilde{A}(\tau, \theta, \bar{\theta}) = (E(\tau, \theta, \bar{\theta}), F(\tau, \theta, \bar{\theta}), \bar{F}(\tau, \theta, \bar{\theta}))$ as the component multiplet superfields. The superfields $E, F, \bar{F}$ can be expanded in terms of the basic fields $(e, c, \bar{c})$ and auxiliary field (b) along with some extra secondary fields (i.e. $f, \bar{f}, B, g, \bar{g}, s, \bar{s}, \bar{b}$), as given below (see, e.g., [11, 12, 27]):

$$E(\tau, \theta, \bar{\theta}) = e(\tau) + \theta \bar{f}(\tau) + \bar{\theta} f(\tau) + \mathrm{i}\theta\bar{\theta} B(\tau),$$
$$F(\tau, \theta, \bar{\theta}) = c(\tau) + \mathrm{i}\theta \bar{b}(\tau) + \mathrm{i}\bar{\theta} g(\tau) + \mathrm{i}\theta\bar{\theta} s(\tau),$$
$$\bar{F}(\tau, \theta, \bar{\theta}) = \bar{c}(\tau) + \mathrm{i}\theta \bar{g}(\tau) + \mathrm{i}\bar{\theta} b(\tau) + \mathrm{i}\theta\bar{\theta} \bar{s}(\tau). \tag{3.1}$$

It is straightforward to note that the local fields $f(\tau)$, $\bar{f}(\tau)$, $c(\tau)$, $\bar{c}(\tau)$, $s(\tau)$, $\bar{s}(\tau)$ on the RHS are fermionic (anticommuting) in nature and the bosonic (commuting) local fields in (3.1) are $e(\tau)$, $B(\tau)$, $g(\tau)$, $\bar{g}(\tau)$, $b(\tau)$, $\bar{b}(\tau)$. It is evident that, in the above expansion, the bosonic and fermionic degrees of freedom match. This requirement is essential for the validity and sanctity of any arbitrary supersymmetric theory in the superfield formulation. In fact, all the secondary fields will be expressed in terms of basic fields due to the restrictions emerging from the application of the horizontality condition (see, e.g., [11, 12, 27])

$$\tilde{d}\tilde{V} = dA = 0, \quad d = d\tau \partial_\tau, \quad A = d\tau e(\tau), \quad d^2 = 0, \tag{3.2}$$

where the super exterior derivative $\tilde{d}$ and the super connection 1-form $\tilde{V}$ are defined as

$$\tilde{d} = d\tau\partial_\tau + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}, \qquad (3.3)$$
$$\tilde{V} = d\tau E(\tau,\theta,\bar{\theta}) + d\theta\bar{F}(\tau,\theta,\bar{\theta}) + d\bar{\theta}F(\tau,\theta,\bar{\theta}).$$

It will be noted that super 1-form connection $\tilde{V}$ is overall bosonic in nature because of the fact that the superfields E and $(F, \bar{F})$ are bosonic ($E^2 \neq 0$) and fermionic ($F^2 = \bar{F}^2 = 0$), respectively. They have been combined together with $d\tau$ and $(d\theta, d\bar{\theta})$ in such a fashion that $\tilde{V}$ becomes bosonic. We expand $\tilde{d}\tilde{V}$, present in the LHS of (3.2), as

$$\begin{aligned}\tilde{d}\tilde{V} &= (d\tau \wedge d\theta)(\partial_\tau\bar{F} - \partial_\theta E) - (d\theta \wedge d\theta)(\partial_\theta \bar{F}) \\ &+ (d\tau \wedge d\bar{\theta})(\partial_\tau F - \partial_{\bar{\theta}} E) \qquad (3.4)\\ &- (d\theta \wedge d\bar{\theta})(\partial_\theta F + \partial_{\bar{\theta}}\bar{F}) - (d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}} F).\end{aligned}$$

Ultimately, the application of the horizontality condition ($\tilde{d}\tilde{V} = dA = 0$) yields

$$f(\tau) = \dot{c}(\tau), \quad \bar{f}(\tau) = \dot{\bar{c}}(\tau), \quad s(x) = \bar{s}(x) = 0, \qquad (3.5)$$
$$B(\tau) = \dot{b}(\tau), \quad b(\tau) + \bar{b}(\tau) = 0, \quad g(\tau) = \bar{g}(\tau) = 0.$$

It may be emphasized that the gauge (einbein) field $e(\tau)$ is a scalar potential depending only on a single variable parameter τ. This is why the curvature is zero (i.e. $dA = 0$) because $d\tau \wedge d\tau = 0$[4]. The insertion of all the above values (cf. (3.5)) in the expansion (3.1) leads to the derivation of the (anti-) BRST symmetry transformations for the gauge and (anti-) ghost fields of the theory. This statement can be expressed, in an explicit form, as given below:

$$\begin{aligned}E(\tau,\theta,\bar{\theta}) &= e(\tau) + \theta\dot{\bar{c}}(\tau) + \bar{\theta}\dot{c}(\tau)) + \mathrm{i}\theta\bar{\theta}\dot{b}(\tau), \\ F(\tau,\theta,\bar{\theta}) &= c(\tau) - \mathrm{i}\theta b(\tau), \\ \bar{F}(\tau,\theta,\bar{\theta}) &= \bar{c}(\tau) + \mathrm{i}\bar{\theta} b(\tau). \qquad (3.6)\end{aligned}$$

In addition, this exercise provides the physical interpretation for the (anti-) BRST charges $Q^{(1)}_{(a)b}$ as the generators (cf. (2.15)) of translations (i.e. $\mathrm{Lim}_{\bar{\theta}\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta})$) along the Grassmannian directions of the supermanifold. Both these observations can be succinctly expressed, in a combined way, by re-writing the super expansion (3.1) as[5]

$$\begin{aligned}E(\tau,\theta,\bar{\theta}) &= e(\tau) + \theta\left(s^{(1)}_{ab} e(\tau)\right) + \bar{\theta}\left(s^{(1)}_{b} e(\tau)\right) \\ &+ \theta\bar{\theta}\left(s^{(1)}_{b} s^{(1)}_{ab} e(\tau)\right), \\ F(\tau,\theta,\bar{\theta}) &= c(\tau) + \theta\left(s^{(1)}_{ab} c(\tau)\right) + \bar{\theta}\left(s^{(1)}_{b} c(\tau)\right) \\ &+ \theta\bar{\theta}\left(s^{(1)}_{b} s^{(1)}_{ab} (\tau)\right), \\ \bar{F}(\tau,\theta,\bar{\theta}) &= \bar{c}(\tau) + \theta\left(s^{(1)}_{ab} \bar{c}(\tau)\right) + \bar{\theta}\left(s^{(1)}_{b} \bar{c}(\tau)\right) \\ &+ \theta\bar{\theta}\left(s^{(1)}_{b} s^{(1)}_{ab} \bar{c}(\tau)\right). \qquad (3.7)\end{aligned}$$

It should be noted that the third and fourth terms of the expansion for F and the second and fourth terms of the expansion for $\bar{F}$ are zero because $(s^{(1)}_b c = 0, s^{(1)}_{ab}\bar{c} = 0)$.

Let us concentrate on the derivation of the nilpotent (i.e. $(s^{(1)}_{(a)b})^2 = 0$) (anti-) BRST symmetry transformations $s^{(1)}_{(a)b}$ for the Lorentz scalar fields $\chi(\tau), \psi_5(\tau)$ and the Lorentz vector target fields $(x_\mu(\tau), \psi_\mu(\tau), p_\mu(\tau))$. In the derivation of the nilpotent transformations for these fields, under the framework of augmented superfield formalism, it is the invariance of the conserved quantities on the (super) manifolds that plays a key role. However, at times, one has to tap the inputs from the super expansions (3.6), derived after the application of the horizontality condition, for the precise derivations of the nilpotent transformations. Thus, to be very precise, it is the interplay of the horizontality condition and the invariance of the conserved charges that enables us to derive the nilpotent (anti-) BRST transformations. To justify this assertion, first of all, we start off with the super expansion of the superfields $(X^\mu, P_\mu)(\tau,\theta,\bar{\theta})$), corresponding to the ordinary target variables $(x^\mu, p_\mu)(\tau)$ (that specify the Minkowski cotangent manifold and are present in the first-order Lagrangian $L^{(m)}_{\mathrm{f}}$), as

$$\begin{aligned}X_\mu(\tau,\theta,\bar{\theta}) &= x_\mu(\tau) + \theta\bar{R}_\mu(\tau) + \bar{\theta}R_\mu(\tau) + \mathrm{i}\theta\bar{\theta}S_\mu(\tau), \\ P_\mu(\tau,\theta,\bar{\theta}) &= p_\mu(\tau) + \theta\bar{F}_\mu(\tau) + \bar{\theta}F_\mu(\tau) + \mathrm{i}\theta\bar{\theta}T_\mu(\tau).\end{aligned} \qquad (3.8)$$

It is evident that, in the limit $(\theta,\bar{\theta}) \to 0$, we get back the canonically conjugate target space variables $(x^\mu(\tau), p_\mu(\tau))$ of the first-order Lagrangian $L^{(m)}_{\mathrm{f}}$ in (2.1). Furthermore, the number of bosonic fields $(x_\mu, p_\mu, S_\mu, T_\mu)$ do match with the fermionic fields $(F_\mu, \bar{F}_\mu, R_\mu, \bar{R}_\mu)$ so that the above expansion becomes consistent with the basic tenets of supersymmetry. All the component fields on the RHS of the expansion (3.8) are functions of the monotonically increasing parameter τ of the world-line. As emphasized in Sect. 2, three most decisive features of the free relativistic particle are

(i) $\dot{p}_\mu = 0$,
(ii) $p \cdot \psi - m\psi_5 = 0$, and
(iii) $p^2 - m^2 = 0$.

To be very specific, it can be seen that the conserved *gauge* charge $Q_g = \frac{1}{2}\left(p^2 - m^2\right)$ couples to the "gauge" (einbein) field $e(\tau)$ in the Lagrangian $L^{(m)}_{\mathrm{f}}$ to maintain the

[4] It is interesting to point out that, unlike the above case, for the 1-form ($A = dx^\mu A_\mu$) Abelian gauge theory, where the gauge field is a vector potential $A_\mu(x)$, the 2-form curvature $dA = \frac{1}{2}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ is not equal to zero and it defines the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the Abelian gauge theory. For the 1-form non-Abelian gauge theory, the Maurer–Cartan equation $F = dA + A \wedge A$ defines the 2-form F which, in turn, leads to the derivation of the corresponding group valued field strength tensor $F_{\mu\nu}$.

[5] It is worthwhile to note that the anti-BRST transformations $s^{(1)}_{ab}$ for the system, described by the Lagrangian in (2.12), are $s^{(1)}_{ab} x_\mu = \bar{c}p_\mu, s^{(1)}_{ab}\bar{c} = 0, s^{(1)}_{ab} p_\mu = 0, s^{(1)}_{ab} c = -\mathrm{i}b, s^{(1)}_{ab} b = 0, s^{(1)}_{ab}\chi = 0, s^{(1)}_{ab}\psi_5 = 0, s^{(1)}_{ab}\psi_\mu = 0, s^{(1)}_{ab} e = \dot{\bar{c}}$. The key point, that should be emphasized, is the minus sign in $s^{(1)}_{ab} c = -\mathrm{i}b$.

local gauge invariance[6] under the transformations (2.3). For the BRST invariant Lagrangian (2.12), the same kind of coupling exists for the local BRST invariance to be maintained in the theory. The invariance of the mass-shell condition ($p^2 - m^2 = 0$) (i.e. a conserved and gauge invariant quantity) as well as the conservation of the gauge invariant momenta ($\dot p_\mu = 0$) on the (super) manifolds, namely

$$P_\mu(\tau,\theta,\bar\theta) P^\mu(\tau,\theta,\bar\theta) - m^2 = p_\mu(\tau) p^\mu(\tau) - m^2,$$
$$\dot P_\mu(\tau,\theta,\bar\theta) = \dot p_\mu(\tau), \tag{3.9}$$

imply the following restrictions:

$$F_\mu(\tau) = \bar F_\mu(\tau) = T_\mu(\tau) = 0, \; P_\mu(\tau,\theta,\bar\theta) = p_\mu(\tau). \tag{3.10}$$

In other words, the invariance of the mass-shell condition as well as the conserved momenta on the (super) manifolds enforces $P_\mu(\tau,\theta,\bar\theta)$ to be independent of the Grassmannian variables θ and $\bar\theta$. To be consistent with our earlier interpretations for the (anti-) BRST charges, in the language of translation generators along the Grassmannian directions $(\theta)\bar\theta$ of the supermanifold, it can be seen that the above equation can be re-expressed as

$$P_\mu(\tau,\theta,\bar\theta) = p_\mu(\tau) + \theta\left(s_{ab}^{(1)} p_\mu(\tau)\right) + \bar\theta\left(s_b^{(1)} p_\mu(\tau)\right)$$
$$+\theta\bar\theta\left(s_b^{(1)} s_{ab}^{(1)} p_\mu(\tau)\right). \tag{3.11}$$

The above equation, vis-à-vis (3.10), makes it clear that $s_b^{(1)} p_\mu(\tau) = 0$ and $s_{ab}^{(1)} p_\mu(\tau) = 0$.

Before we shall derive the nilpotent (anti-) BRST transformations for $x_\mu(\tau)$, it is useful to compute these transformations for the fermionic gauge field $\chi(\tau)$ and other fields $\psi_5(\tau)$ as well as $\psi_\mu(\tau)$. With this end in mind, let us have the super expansions for the superfields, corresponding to these fields, as follows:

$$K(\tau,\theta,\bar\theta) = \chi(\tau) + \theta\bar b_1(\tau) + \bar\theta b_1(\tau) + \theta\bar\theta f_1(\tau),$$
$$\Psi_5(\tau,\theta,\bar\theta) = \psi_5(\tau) + \theta\bar B_5(\tau) + \bar\theta B_5(\tau) + \theta\bar\theta f_5(\tau),$$
$$\Psi_\mu(\tau,\theta,\bar\theta) = \psi_\mu(\tau) + \theta\bar b_\mu(\tau) + \bar\theta b_\mu(\tau) + \theta\bar\theta f_\mu(\tau). \tag{3.12}$$

It will be noted that, in the limit $(\theta,\bar\theta) \to 0$, we do obtain the usual local fields $\chi(\tau), \psi_5(\tau)$ and $\psi_\mu(\tau)$ and the fermionic $(\chi, \psi_5, \psi_\mu, f_\mu, f_1, f_5)$ and bosonic $(b_1, \bar b_1, B_5, \bar B_5, b_\mu, \bar b_\mu)$ degrees of freedom do match in the above expansion. Let us focus on the conserved quantities $\dot\chi = 0, \ddot\psi_5 = \dot\chi m = 0, \ddot\psi_\mu = \dot\chi p_\mu + \chi \dot p_\mu = 0$ (cf. (2.10)). The invariance of $\dot\chi(\tau) = 0$ on the (super) manifolds[7] leads to the following consequences:

$$\dot K(\tau,\theta,\bar\theta) = \dot\chi(\tau) = 0 \quad \Rightarrow \quad \dot b_1 = \dot{\bar b}_1 = \dot f_1 = 0. \tag{3.13}$$

One of the solutions is $b_1 = \bar b_1 = f_1 = C$ where C is a τ-independent constant. Geometrically, this amounts to the shift of the superfield $K(\tau,\theta,\bar\theta)$ along the θ- and $\bar\theta$-directions by a constant value C. One can choose the θ and $\bar\theta$ axes on the supermanifold in such a manner that this constant C is zero. Thus, ultimately, we obtain $b_1 = \bar b_1 = f_1 = 0$. Interpreted in the light of (3.6), (3.7) and (3.11), this shows that $s_{(a)b}^{(1)}\chi = 0$ (i.e. $K(\tau,\theta,\bar\theta) = \chi(\tau)$). As a side remark, it is worthwhile to mention that the above explicit equality (3.13) can be re-expressed as the equality of the conserved quantities *only*. In other words, the restriction $K(\tau,\theta,\bar\theta) = \chi(\tau)$, directly implies that $b_1 = \bar b_1 = f_1 = 0$. For the (non-) Abelian gauge theories, such kind of equality has been taken into account [24–26] where only the expression for the conserved quantity has been equated on the (super) manifolds. In view of the above, it can be seen that the following invariances of the conserved quantities (cf. (2.10) for details) on the (super) manifolds:

$$\dot\Psi_5(\tau,\theta,\bar\theta) - K(\tau,\theta,\bar\theta) m = \dot\psi_5(\tau) - \chi(\tau) m,$$
$$\dot\Psi_\mu(\tau,\theta,\bar\theta) - K(\tau,\theta,\bar\theta) P_\mu(\tau,\theta,\bar\theta)$$
$$= \dot\psi_\mu(\tau) - \chi(\tau) p_\mu(\tau), \tag{3.14}$$

imply the following restrictions for the expansion in (3.12), namely

$$\bar B_5 = B_5 = f_5 = 0, \quad \bar b_\mu = b_\mu = f_\mu = 0. \tag{3.15}$$

In the above derivation, we have exploited the results of (3.10) (i.e. $P_\mu(\tau,\theta,\bar\theta) = p_\mu(\tau)$) and (3.13) (i.e. $K(\tau,\theta,\bar\theta) = \chi(\tau)$). Insertions of (3.15) into (3.12) imply that $s_{(a)b}^{(1)}\psi_5 = 0$ (i.e. $\Psi_5(\tau,\theta,\bar\theta) = \psi_5(\tau)$) and $s_{(a)b}^{(1)}\psi_\mu = 0$ (i.e. $\Psi_\mu(\tau,\theta,\bar\theta) = \psi_\mu(\tau)$). It is worthwhile to emphasize that the above solutions are one set of the *simplest* solutions which are of interest to us. A more general solution (than the above) might exist.

Now the stage is set for the derivation of the nilpotent (anti-) BRST transformations for the target space coordinate variable $x_\mu(\tau)$. One of the most important relations, that plays a pivotal role in the derivation of the mass-shell condition ($p^2 - m^2 = 0$) for the Lagrangian $L_s^{(m)}$, is $\dot x_\mu(\tau) = e(\tau) p_\mu(\tau) - i\chi\psi_\mu$. This is due to the fact that $(\partial L_s^{(m)}/\partial e) = 0$ implies that $e^2 m^2 = (\dot x_\mu + i\chi\psi_\mu)^2$ and $p_\mu = \left(\partial L_s^{(m)}/\partial \dot x^\mu\right) = e^{-1}(\dot x_\mu + i\chi\psi_\mu)$. A simple way to derive the (anti-) BRST transformations for the coordinate target variable $x_\mu(\tau)$ is to require the invariance of this central relation on the (super) manifolds, as

[6] Exactly the same kind of gauge coupling exists between the Dirac fields for the fermions (electrons, positrons, quarks, etc.) and the gauge boson field of the *interacting* 1-form (non-) Abelian gauge theories where the matter conserved current $J_\mu = \bar\psi\gamma_\mu\psi$, constructed by the Dirac fields, couples to the gauge field A_μ of the (non-) Abelian gauge theories to maintain the local gauge invariance (see, e.g., [32]).

[7] This condition, in a more sophisticated language, is just the gauge choice for the system.

$$
\begin{aligned}
&\dot{X}_\mu(\tau,\theta,\bar\theta) - E(\tau,\theta,\bar\theta)P_\mu(\tau,\theta,\bar\theta)\\
&\quad + \mathrm{i}K(\tau,\theta,\bar\theta)\Psi_\mu(\tau,\theta,\bar\theta) \qquad (3.16)\\
&\quad = \dot{x}_\mu(\tau) - e(\tau)p_\mu(\tau) + \mathrm{i}\chi(\tau)\psi_\mu(\tau),
\end{aligned}
$$

where $E(\tau,\theta,\bar\theta)$ is the expansion in (3.6) which has been obtained after the application of the horizontality condition. Exploiting the relations $P_\mu(\tau,\theta,\bar\theta) = p_\mu(\tau)$ from (3.10) and $K(\tau,\theta,\bar\theta) = \chi(\tau)$, it can be seen that the following relations emerge from (3.16):

$$
\dot{\bar R}_\mu = \dot{\bar c} p_\mu, \quad \dot{R}_\mu = \dot{c} p_\mu, \quad \dot{S}_\mu = \dot{b} p_\mu. \qquad (3.17)
$$

At this crucial stage, we summon one of the most decisive physical insights into the characteristic features of a free spinning relativistic particle which states that there is no action of any kind of force (i.e. $\dot{p}_\mu(\tau) = 0$) on the *free* motion of the particle. Having taken into account this decisive input, we obtain from (3.17), the following relations:

$$
\begin{aligned}
&\dot{\bar R}_\mu \equiv \partial_\tau \bar R_\mu = \partial_\tau(\bar c p_\mu), \quad \dot R_\mu \equiv \partial_\tau R_\mu = \partial_\tau(c p_\mu),\\
&\dot S_\mu \equiv \partial_\tau S_\mu = \partial_\tau(b p_\mu), \qquad (3.18)
\end{aligned}
$$

which lead to

$$
\bar R_\mu(\tau) = \bar c p_\mu, \quad R_\mu(\tau) = c p_\mu, \quad S_\mu(\tau) = b p_\mu. \qquad (3.19)
$$

The insertions of these values into the expansion (3.8) lead to the derivation of the nilpotent (anti-) BRST transformations ($s^{(1)}_{(a)b}$) on the target space coordinate field $x_\mu(\tau)$:

$$
\begin{aligned}
X_\mu(\tau,\theta,\bar\theta) &= x_\mu(\tau) + \theta\left(s^{(1)}_{ab} x_\mu(\tau)\right) + \bar\theta\left(s^{(1)}_{b} x_\mu(\tau)\right)\\
&\quad + \theta\bar\theta\left(s^{(1)}_{b} s^{(1)}_{ab} x_\mu(\tau)\right). \qquad (3.20)
\end{aligned}
$$

In our recent papers [24–26] on interacting 1-form (non-) Abelian gauge theories, it has been shown that there is a beautiful consistency and complementarity between the horizontality condition and the requirement of the invariance of conserved matter (super) currents on the (super) manifolds. The former restriction leads to the derivation of nilpotent symmetries for the gauge and (anti-) ghost fields. The latter restriction yields such kind of transformations for the matter fields. For the case of the free spinning relativistic particle, it can be seen that the invariance of the gauge invariant and conserved quantities on the (super) manifolds, leads to the derivation of the transformations for the target field variables. To corroborate this assertion, we observe that the conserved and gauge invariant charge $Q_g = \frac{1}{2}(p^2 - m^2)$ is the analogue of the conserved *matter* current of the 1-form interacting (non-) Abelian gauge theory. Since the expansion for $P_\mu(x.\theta,\bar\theta)$ is trivial (cf. (3.10)), we have to re-express the mass-shell condition (i.e. $e^2(p^2 - m^2) = (\dot x_\mu + \mathrm{i}\chi\psi_\mu)^2 - e^2m^2$) in the language of the superfields (3.6) and $\Psi_\mu(\tau,\theta,\bar\theta) = \psi_\mu(\tau), K(\tau,\theta,\bar\theta) = \chi(\tau)$. Thus, the invariance of the conserved (super) charges on the (super) manifolds is

$$
\begin{aligned}
&[\dot X_\mu(\tau,\theta,\bar\theta) + \mathrm{i}K(\tau,\theta,\bar\theta)\Psi_\mu(\tau,\theta,\bar\theta)]\\
&\quad \times[\dot X^\mu(\tau,\theta,\bar\theta) + \mathrm{i}K(\tau,\theta,\bar\theta)\Psi^\mu(\tau,\theta,\bar\theta)]\\
&\quad - m^2 E(\tau,\theta,\bar\theta)E(\tau,\theta,\bar\theta)\\
&\quad = [\dot x_\mu(\tau) + \mathrm{i}\chi(\tau)\psi_\mu(\tau)][\dot x^\mu(\tau) + \mathrm{i}\chi(\tau)\psi^\mu(\tau)]\\
&\qquad - m^2e^2. \qquad (3.21)
\end{aligned}
$$

The equality of the appropriate terms from the LHS and RHS leads to

$$
\begin{aligned}
&(\dot x_\mu + \mathrm{i}\chi\psi_\mu)\,\dot{\bar R}^\mu = m^2 e\dot{\bar c}, \quad (\dot x_\mu + \mathrm{i}\chi\psi_\mu)\,\dot R^\mu = m^2 e\dot c,\\
&(\dot x_\mu + \mathrm{i}\chi\psi_\mu)\,\dot S^\mu = m^2 e\dot b, \quad \dot R_\mu\,\dot{\bar R}^\mu = m^2\dot c\dot{\bar c}. \qquad (3.22)
\end{aligned}
$$

With the help of the key relation $\dot x_\mu + \mathrm{i}\chi\psi_\mu = e p_\mu$, we obtain the expressions for $\dot R_\mu, \dot{\bar R}_\mu, \dot S_\mu$, exactly the same as the ones given in (3.17) *for the mass-shell condition* $p^2 - m^2 = 0$ *to be valid*. Exploiting the no force (i.e. $\dot p_\mu = 0$) criterion on the free motion of a spinning relativistic particle, we obtain the expressions for $R_\mu, \bar R_\mu, S_\mu$ in exactly the same form as given in (3.19). The insertion of these values in (3.8) leads to the same expansion as given in (3.20). This provides the geometrical interpretation for the (anti-) BRST charges as the *translational generators*. It should be noted that the restrictions in (3.16) and (3.21) are intertwined. However, the latter is more physical because it states the invariance of the conserved and gauge invariant *mass-shell* condition *explicitly*. As a side remark, we would like to comment on the other conserved quantity $p\cdot\psi - m\psi_5 = 0$. It is straightforward to check that this conserved quantity is automatically satisfied on the (super) manifolds due to the fact that $\Psi_5(\tau,\theta,\bar\theta) = \psi_5(\tau), \Psi_\mu(\tau,\theta,\bar\theta) = \psi_\mu(\tau), P_\mu(\tau,\theta,\bar\theta) = p_\mu(\tau)$. In fact, the results $\Psi_5(\tau,\theta,\bar\theta) = \psi_5(\tau), \Psi_\mu(\tau,\theta,\bar\theta) = \psi_\mu(\tau)$ can be obtained from this very conserved quantity if we follow the same trick as we have exploited, in the above, for the conserved and gauge invariant quantity $p^2 - m^2 = 0$ for the derivation of $s^{(1)}_{(a)b}$ for $x_\mu(\tau)$.

4 Supergauge BRST symmetries: augmented superfield formalism

We derive here the nilpotent ($(s^{(2)}_{(a)b})^2 = 0$) (anti-) BRST symmetry transformations $s^{(2)}_{(a)b}$ (cf. (2.13)), corresponding to the supergauge transformations in (2.4), in the framework of the augmented superfield formalism. The crucial assumption here is the condition $\beta^2 = 0$ which can be satisfied if and only if the bosonic (anti-) ghost fields $(\bar\beta)\beta$ were made up of two fermionic ghost fields. In contrast to the super 1-form *bosonic* connection $\tilde V$, quoted in (3.3), we define here a fermionic super 1-form connection $\tilde{\mathcal F}$ as follows:

$$
\begin{aligned}
\tilde{\mathcal F} &= dZ(\tilde F) \qquad (4.1)\\
&= d\tau K(\tau,\theta,\bar\theta) + \mathrm{i}d\theta\bar{\mathcal B}(\tau,\theta,\bar\theta) + \mathrm{i}d\bar\theta\mathcal B(\tau,\theta,\bar\theta),
\end{aligned}
$$

where the supermultiplet

$$\tilde{F}(\tau,\theta,\bar{\theta}) = (K(\tau,\theta,\bar{\theta}), \mathrm{i}\mathcal{B}(\tau,\theta,\bar{\theta}), \mathrm{i}\bar{\mathcal{B}}(\tau,\theta,\bar{\theta}))$$

has three superfields and the fermionic super gauge field $K(\tau,\theta,\bar{\theta})$ has an expansion as given in (3.12). The bosonic superfields $\mathcal{B}(\tau,\theta,\bar{\theta})$ and $\bar{\mathcal{B}}(\tau,\theta,\bar{\theta})$ have the following expansions:

$$\mathcal{B}(\tau,\theta,\bar{\theta}) = \beta(\tau) + \mathrm{i}\theta\bar{f}_2(\tau) + \mathrm{i}\bar{\theta}f_3(\tau) + \mathrm{i}\theta\bar{\theta}b_2(\tau),$$
$$\bar{\mathcal{B}}(\tau,\theta,\bar{\theta}) = \bar{\beta}(\tau) + \mathrm{i}\theta\bar{f}_3(\tau) + \mathrm{i}\bar{\theta}f_2(\tau) + \mathrm{i}\theta\bar{\theta}\bar{b}_2(\tau), \quad (4.2)$$

which yield the bosonic (anti-) ghost fields $(\bar{\beta})\beta$ in the limit $(\theta,\bar{\theta}) \to 0$ and the bosonic $(\beta,\bar{\beta},b_2,\bar{b}_2)$ and fermionic $(f_2,\bar{f}_2,f_3,\bar{f}_3)$ degrees of freedom do match. The requirement of the horizontality condition (with the super exterior derivative $\tilde{d}$ defined in (3.3)):

$$\tilde{d}\tilde{\mathcal{F}} = d\mathcal{F} = 0, \quad d = d\tau\partial_\tau, \quad \mathcal{F} = d\tau\chi(\tau), \quad d^2 = 0, \quad (4.3)$$

leads to the derivation of the secondary fields in terms of the basic fields as well as the auxiliary fields. In fact, the explicit expression for $\tilde{d}\tilde{\mathcal{F}}$ is

$$\tilde{d}\tilde{\mathcal{F}} = (d\tau\wedge d\theta)(\mathrm{i}\partial_\tau\bar{\mathcal{B}} - \partial_\theta K) - \mathrm{i}(d\theta\wedge d\theta)(\partial_\theta\bar{\mathcal{B}})$$
$$+(d\tau\wedge d\bar{\theta})(\mathrm{i}\partial_\tau\mathcal{B} - \partial_{\bar{\theta}}K) \quad (4.4)$$
$$-\mathrm{i}(d\theta\wedge d\bar{\theta})(\partial_\theta\mathcal{B} + \partial_{\bar{\theta}}\bar{\mathcal{B}}) - \mathrm{i}(d\bar{\theta}\wedge d\bar{\theta})(\partial_{\bar{\theta}}\mathcal{B}).$$

The application of the horizontality condition $\tilde{d}\tilde{\mathcal{F}} = d\mathcal{F} = 0$ leads to the following relations:

$$\partial_\theta\bar{\mathcal{B}} = 0, \quad \partial_{\bar{\theta}}\mathcal{B} = 0, \quad \partial_{\bar{\theta}}\bar{\mathcal{B}} + \partial_\theta\mathcal{B} = 0,$$
$$\partial_\theta K = \mathrm{i}\partial_\tau\bar{\mathcal{B}}, \quad \partial_{\bar{\theta}}K = \mathrm{i}\partial_\tau\mathcal{B}. \quad (4.5)$$

The first two relations, in the above, produce $b_2 = \bar{b}_2 = 0, f_3 = \bar{f}_3 = 0$. The third one leads to $f_2 + \bar{f}_2 = 0$. Choosing $f_2 = \gamma$ implies that $\bar{f}_2 = -\gamma$ and the bosonic superfields $\mathcal{B}$ and $\bar{\mathcal{B}}$ become chiral and anti-chiral superfields, respectively, with the following expansions:

$$\mathcal{B}(\tau,\theta) = \beta(\tau) - \mathrm{i}\theta\gamma \equiv \beta(\tau) - \theta\left(s_{ab}^{(2)}\beta(\tau)\right),$$
$$\bar{\mathcal{B}}(\tau,\bar{\theta}) = \bar{\beta}(\tau) + \mathrm{i}\theta\gamma \equiv \bar{\beta}(\tau) + \bar{\theta}\left(s_b^{(2)}\bar{\beta}(\tau)\right). \quad (4.6)$$

It will be noted that the Nakanishi–Lautrup fermionic $(\gamma^2 = 0)$ auxiliary field $\gamma(\tau)$ is *not* a basic dynamical field variable of the theory (cf. (2.14)). The above equations establish that $s_b^{(2)}\beta(\tau) = 0$ and $s_{ab}^{(2)}\bar{\beta}(\tau) = 0$. Exploiting the expressions for $\mathcal{B}(\tau,\theta)$ and $\bar{\mathcal{B}}(\tau,\bar{\theta})$ (cf. (4.6)) in the last two relations of (4.5), we obtain the following values for the component local secondary fields of the expansion for $K(\tau,\theta,\bar{\theta})$ in (3.12):

$$b_1(\tau) = \mathrm{i}\dot{\beta}(\tau), \quad \bar{b}_1(\tau) = \mathrm{i}\dot{\bar{\beta}}(\tau), \quad f_1(\tau) = -\dot{\gamma}(\tau). \quad (4.7)$$

The insertions of the above values in the expansion of $K(\tau,\theta,\bar{\theta})$ yields

$$K(\tau,\theta,\bar{\theta}) = \chi(\tau) + \theta\left(s_{ab}^{(2)}\chi(\tau)\right) + \bar{\theta}\left(s_b^{(2)}\chi(\tau)\right)$$
$$+\theta\bar{\theta}\left(s_b^{(2)}s_{ab}^{(2)}\chi(\tau)\right). \quad (4.8)$$

It should be pointed out that the expressions in (4.6) for the bosonic superfields can also be written in an exactly the same form as (4.8) as $s_b^{(2)}\beta(\tau) = 0$ and $s_{ab}^{(2)}\bar{\beta}(\tau) = 0$. There is only one caveat, however. This has to do with the $(-)$ sign in the expansion of $B(\tau,\theta)$[8]. We shall dwell on it, in detail, in the conclusions part, Sect. 5, of our present paper. It is clear that this exercise provides the geometrical interpretation for the (anti-) BRST charges $Q_{(a)b}^{(2)}$ as the generators (cf. (2.15)) of translations (i.e. $\mathrm{Lim}_{\bar{\theta}\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta})$) along the Grassmannian directions $(\theta)\bar{\theta}$ of the three (1+2)-dimensional supermanifold.

Let us focus on the derivation of the nilpotent transformations $s_{(a)b}^{(2)}$ for the target field variables $(x_\mu(\tau), p_\mu(\tau), \psi_\mu(\tau))$ and the Lorentz scalar fermionic field $\psi_5(\tau)$ in the framework of augmented superfield formalism. Here, once again, the interplay of the horizontality condition and the invariance of the conserved quantities on the (super) manifolds do play a very important and decisive roles. To see it clearly, let us first concentrate on the invariance of the conserved quantities (given in (3.14)) on the (super) manifolds. The explicit substitutions of super expansions yield the following relationships:

$$(\dot{\psi}_5 + \theta\dot{\bar{B}}_5 + \bar{\theta}\dot{B}_5 + \theta\bar{\theta}\dot{f}_5)$$
$$-(\chi + \mathrm{i}\theta\dot{\bar{\beta}} + \mathrm{i}\bar{\theta}\dot{\beta} - \theta\bar{\theta}\dot{\gamma})m$$
$$= \dot{\psi}_5 - \chi m,$$
$$(\dot{\psi}_\mu + \theta\dot{\bar{b}}_\mu + \bar{\theta}\dot{b}_\mu + \theta\bar{\theta}\dot{f}_\mu)$$
$$-(\chi + \mathrm{i}\theta\dot{\bar{\beta}} + \mathrm{i}\bar{\theta}\dot{\beta} - \theta\bar{\theta}\dot{\gamma})p_\mu$$
$$= \dot{\psi}_\mu - \chi p_\mu, \quad (4.9)$$

where we have exploited the expansions of $\Psi_\mu(\tau,\theta,\bar{\theta})$ and $\Psi_5(\tau,\theta,\bar{\theta})$ given in (3.12) and have used the expansion of $K(\tau,\theta,\bar{\theta})$ from (4.8) that has been obtained after the application of the horizontality condition. It is evident that the following relations emerge between the secondary component fields and basic fields:

[8] It will be noted that the explicit form of the anti-BRST transformations $s_{ab}^{(2)}$ for the system are $s_{ab}^{(2)}x_\mu = \bar{\beta}\psi_\mu, s_{ab}^{(2)}p_\mu = 0, s_{ab}^{(2)}\bar{\beta} = 0, s_{ab}^{(2)}\beta = +\mathrm{i}\gamma, s_{ab}^{(2)}\gamma = 0, s_{ab}^{(2)}\psi_\mu = \mathrm{i}\bar{\beta}p_\mu, s_{ab}^{(2)}\chi = \mathrm{i}\dot{\bar{\beta}}, s_{ab}^{(2)}e = 2\bar{\beta}\chi$. The key point that should be emphasized is the fact that, the bosonic nature of the (anti-) ghost fields $(\bar{\beta})\beta$ does *not* allow for a change of sign between $s_{ab}^{(2)}\beta(= +\mathrm{i}\gamma)$ and $s_b^{(2)}\bar{\beta}(= +\mathrm{i}\gamma)$ for the symmetry of the Lagrangian to be maintained. This situation is totally opposite to the case of $s_{(a)b}^{(1)}$ (cf. (2.11)) where the (anti-) ghost fields are fermionic in nature, and, that is why, the change of sign in $s_{ab}^{(1)}c = -\mathrm{i}b, s_b^{(1)}\bar{c} = \mathrm{i}b$ is required.

$$\dot{B}_5 = \mathrm{i}\dot{\beta}m \equiv \partial_\tau(\mathrm{i}\beta m), \quad \dot{\bar{B}}_5 = \mathrm{i}\dot{\bar{\beta}}m \equiv \partial_\tau(\mathrm{i}\bar{\beta}m), \quad (4.10)$$
$$\dot{f}_5 = -\dot{\gamma}m \equiv -\partial_\tau(\gamma m), \quad \dot{\bar{b}}_\mu = \mathrm{i}\dot{\bar{\beta}}p_\mu \equiv \partial_\tau(\mathrm{i}\bar{\beta}p_\mu),$$
$$\dot{b}_\mu = \mathrm{i}\dot{\beta}p_\mu \equiv \partial_\tau(\mathrm{i}\beta p_\mu), \quad \dot{f}_\mu = -\dot{\gamma}p_\mu \equiv -\partial_\tau(\gamma p_\mu),$$

where, in the latter set of entries, we have used the requirement of the free motion ($\dot{p}_\mu = 0$) of a free spinning relativistic particle. Ultimately, the insertions of the above values in the expansions (3.12), yields the following expansions in terms of $s^{(2)}_{(a)b}$ (cf. (2.13)):

$$\Psi_5(\tau,\theta,\bar{\theta}) = \psi_5(\tau) + \theta\left(s^{(2)}_{ab}\psi_5(\tau)\right) + \bar{\theta}\left(s^{(2)}_{b}\psi_5(\tau)\right) + \theta\bar{\theta}\left(s^{(2)}_{b}s^{(2)}_{ab}\psi_5(\tau)\right),$$
$$\Psi_\mu(\tau,\theta,\bar{\theta}) = \psi_\mu(\tau) + \theta\left(s^{(2)}_{ab}\psi_\mu(\tau)\right) + \bar{\theta}\left(s^{(2)}_{b}\psi_\mu(\tau)\right) + \theta\bar{\theta}\left(s^{(2)}_{b}s^{(2)}_{ab}\psi_\mu(\tau)\right). \quad (4.11)$$

The above expansions produce the same geometrical interpretations for the symmetries $s^{(2)}_{(a)b}$ and the generators $Q^{(2)}_{(a)b}$ (i.e. the translational generators along the Grassmannian directions) as the conclusions drawn for the expansion in (4.8) for $K(\tau,\theta,\bar{\theta})$.

Having obtained the super expansions of superfields (i.e. $K, \mathcal{B}, \bar{\mathcal{B}}, \Psi, \Psi_5$) in terms of the local τ-dependent ordinary basic fields in (4.6), (4.8) and (4.11), the stage is now set for the derivation of the nilpotent symmetry transformations for the einbein field $e(\tau)$ and the canonically conjugate target space field variables $x_\mu(\tau)$ and $p_\mu(\tau)$. It is clear that $\dot{p}_\mu = 0$ and the mass-shell condition $p^2 - m^2 = 0$ are
(i) supergauge invariant (i.e. $\delta_{sg}p_\mu = 0$), and
(ii) conserved quantities. Thus, their invariance on the (super) manifolds, once again, leads to the same conclusions as illustrated in (3.9) and (3.10). As a consequence, we have $s^{(2)}_{(a)b}p_\mu = 0$. Now, the central problem is to obtain the nilpotent transformations for $e(\tau)$ and $x_\mu(\tau)$. In this connection, it turns out that
(i) the intertwined relations given in (3.16) and (3.21), and
(ii) the conserved quantity $p \cdot \psi - m\psi_5 = 0$ (expressed in terms of $p_\mu = e^{-1}(\dot{x}_\mu + \mathrm{i}\chi\psi_\mu)$ so that it becomes $\dot{x} \cdot \psi - me\psi_5 = 0$), emerge to help in the final computation. Exploiting the explicit expressions for the expansions of $\Psi_\mu(\tau,\theta,\bar{\theta})$ and $\Psi_5(\tau,\theta,\bar{\theta})$ given in (4.11) and inserting the values of $\dot{X}_\mu(\tau,\theta,\bar{\theta})$ and $E(\tau,\theta,\bar{\theta})$ from (3.8) and (3.1) in the following invariance of the conserved quantity on the (super) manifolds:

$$\dot{X} \cdot \Psi - mE\Psi_5 = \dot{x} \cdot \psi - me\psi_5, \quad (4.12)$$

we obtain the following relationships:

$$\dot{\bar{R}} \cdot \psi - m\bar{f}\psi_5 = \mathrm{i}e\bar{\beta}m^2 - \mathrm{i}\bar{\beta}(\dot{x} \cdot p),$$
$$\dot{R} \cdot \psi - mf\psi_5 = \mathrm{i}e\beta m^2 - \mathrm{i}\beta(\dot{x} \cdot p),$$
$$\mathrm{i}(\dot{S} \cdot \psi) + \mathrm{i}\bar{\beta}(\dot{R} \cdot p) - \mathrm{i}\beta\left(\dot{\bar{R}} \cdot p\right) + \mathrm{i}\bar{f}\beta m^2 - \mathrm{i}f\bar{\beta}m^2 - \mathrm{i}B\psi_5 = \gamma(\dot{x} \cdot p) - e\gamma m^2. \quad (4.13)$$

Similarly, tapping the potential of the cute relationship in (3.16) (which is finally intertwined with the invariance of the conserved quantity in (3.21)), we obtain the following connections among the component secondary local fields and the basic local fields:

$$\dot{R}_\mu - fp_\mu = \dot{\beta}\psi_\mu - \beta\chi p_\mu,$$
$$\dot{\bar{R}}_\mu - \bar{f}p_\mu = \dot{\bar{\beta}}\psi_\mu - \bar{\beta}\chi p_\mu, \quad (4.14)$$
$$\mathrm{i}\dot{S}_\mu - Bp_\mu = \mathrm{i}\dot{\gamma}\psi_\mu + \mathrm{i}(\dot{\bar{\beta}}\beta - \dot{\beta}\bar{\beta})p_\mu + \mathrm{i}\chi\gamma p_\mu,$$

where the explicit expansions from (3.1) (for E), (3.8) (for X_μ), (4.8) (for K) and (4.11) (for Ψ_μ) have been used. Solution to the above equations on the on-shell (i.e. $\dot{\psi}_\mu = \chi p_\mu, \dot{p}_\mu = 0, \dot{x}_\mu = ep_\mu - \mathrm{i}\chi\psi_\mu, p^2 - m^2 = 0, p \cdot \psi - m\psi_5 = 0$, etc.) are as follows:

$$f = 2\beta\chi, \quad \bar{f} = 2\bar{\beta}\chi, \quad R_\mu = \beta\psi_\mu, \quad \bar{R}_\mu = \bar{\beta}\psi_\mu,$$
$$S_\mu = \gamma\psi_\mu + \beta\bar{\beta}p_\mu, \quad B = 2\dot{\beta}\bar{\beta} + 2\gamma\chi. \quad (4.15)$$

As a side remark, it is interesting to point out that, even from a single relationship in (4.14), some of the above values could be guessed. For instance, the relationship $\dot{R}_\mu - fp_\mu = \dot{\beta}\psi_\mu - \beta\chi p_\mu$ can be re-expressed as $\dot{R}_\mu - fp_\mu = \partial_\tau(\beta\psi_\mu) - \beta\dot{\psi}_\mu - \beta\chi p_\mu$. Exploiting the on-shell condition $\dot{\psi}_\mu = \chi p_\mu$ in the above, it can be seen that $\dot{R}_\mu - fp_\mu = \partial_\tau(\beta\psi_\mu) - 2\beta\chi p_\mu$. This last relation gives a glimpse of $R_\mu = \beta\psi_\mu$ and $f = 2\beta\chi$. In exactly the same manner, it can be seen that $\bar{R}_\mu = \bar{\beta}\psi_\mu$ and $\bar{f} = 2\bar{\beta}\chi$. With these values, other expressions of (4.15) follow, which ultimately, satisfy all the relations derived in (4.13) and (4.14). Insertions of the above values in the expressions for superfield $E(\tau,\theta,\bar{\theta})$ in (3.1) and the superfield $X_\mu(\tau,\theta,\bar{\theta})$ in (3.8) lead to the following:

$$E(\tau,\theta,\bar{\theta}) = e(\tau) + \theta\left(s^{(2)}_{ab}e(\tau)\right) + \bar{\theta}\left(s^{(2)}_{b}e(\tau)\right) + \theta\bar{\theta}\left(s^{(2)}_{b}s^{(2)}_{ab}e(\tau)\right),$$
$$X_\mu(\tau,\theta,\bar{\theta}) = x_\mu(\tau) + \theta\left(s^{(2)}_{ab}x_\mu(\tau)\right) + \bar{\theta}\left(s^{(2)}_{b}x_\mu(\tau)\right) + \theta\bar{\theta}\left(s^{(2)}_{b}s^{(2)}_{ab}x_\mu(\tau)\right). \quad (4.16)$$

The above equation, once again, establishes the fact that the nilpotent (anti-) BRST generators $Q^{(2)}_{(a)b}$ (and corresponding symmetry transformations $s^{(2)}_{(a)b}$) are the translational generators along the Grassmannian directions $(\theta)\bar{\theta}$ of the supermanifolds.

5 Conclusions

In our present endeavour, we have exploited, in an elegant way, the key ideas of the augmented superfield formalism

to derive two sets of anticommuting (i.e. $\{s^{(1)}_{(a)b}, s^{(2)}_{(a)b}\} = 0$) and nilpotent ($(s^{(1,2)}_{(a)b})^2 = 0$) (anti-) BRST symmetry transformations $s^{(1,2)}_{(a)b}$ for *all* the field variables, present in the Lagrangian description of a free massive spinning relativistic particle. The theoretical arsenal of
(i) the horizontality condition, and
(ii) the invariance of the conserved quantities on the (super) manifolds, have played very decisive roles in the above derivations. One of the central new features of our present investigation is the application of the augmented superfield formulation to a supersymmetric system where the bosonic $(\bar\beta)\beta$ and fermionic $(\bar c)c$ (anti-) ghost fields are present together in the (anti-) BRST invariant Lagrangian describing the free motion ($\dot p_\mu = 0$) of the super particle. Of course, for the system under consideration, the reparametrization symmetry invariance and the gauge symmetry invariance are also present. All these symmetries are inter-related.

Even though the pair of bosonic (anti-) ghost fields $(\bar\beta)\beta$ are commutative in nature (i.e. $\beta\bar\beta = \bar\beta\beta, \beta\Sigma = \Sigma\beta, \bar\beta\Sigma = \Sigma\bar\beta$ for the generic field $\Sigma = x_\mu, p_\mu, \psi_\mu, \psi_5, e, \chi, c, \bar c, b, \gamma$), they are taken to be nilpotent of order two ($\beta^2 = 0, \bar\beta^2 = 0$) with the assumption that they are made up of a pair of *fermionic* (anti-) ghost fields (i.e. $\beta \sim c_1 c_2, \bar\beta \sim \bar c_1 \bar c_2, c_1^2 = c_2^2 = 0, c_1 c_2 + c_2 c_1 = 0$, etc.). Such an assumption is essential for a couple of advantageous reasons. First, the (anti-) BRST transformations (3.13) (corresponding to the supergauge symmetry transformations (2.4)) become nilpotent (i.e. $(s^{(2)}_{(a)b})^2 = 0$) under the above assumption. This can be explicitly checked for $(s^{(2)}_{(a)b})^2 x_\mu(\tau) = 0$ and $(s^{(2)}_{(a)b})^2 e(\tau) = 0$ where the conditions $\beta^2 = \bar\beta^2 = 0$ and $\partial_\tau(\beta)^2 = \partial_\tau(\bar\beta)^2 = 0$ are required for the proof of an explicit nilpotency. Second, this assumption also allows $s^{(2)}_{(a)b}$ to decouple from the nilpotent transformations in (2.6) (where, in some sense, they are hidden) and the nilpotent transformations (2.11) so that they could become completely separate and independent. At this stage, it is worth emphasizing that there is no such kind of restriction (i.e. $\beta^2 \neq 0$) on the bosonic ghost field β in the nilpotent transformations listed in (2.6).

In our earlier works [23–27], the horizontality condition was augmented to include the invariance of the conserved matter currents/charges on the (super) manifolds. In our present endeavour, the augmented superfield formalism has been extended to include the invariance of *any* kind of conserved quantities on the (super) manifolds and still
(i) there is a mutual consistency and complementarity between the two above types of restrictions;
(ii) the geometrical interpretations for the nilpotent (anti-) BRST charges $Q^{(1,2)}_{(a)b}$, as the translational generators ($\text{Lim}_{\bar\theta\to 0}(\partial/\partial\theta)$) $\text{Lim}_{\theta\to\, 0}(\partial/\partial\bar\theta)$ along the $(\theta)\bar\theta$-directions of the $(D+2)$-dimensional supermanifold, remains intact;
(iii) the nilpotency of the (anti-) BRST charges $Q^{(1,2)}_{(a)b}$ is encoded in a couple of successive translations (i.e. $(\partial/\partial\theta)^2 = (\partial/\partial\bar\theta)^2 = 0$) along either of the two Grassmannian directions of the supermanifold;
(iv) the anticommutativity of the nilpotent (anti-) BRST charges $Q^{(1,2)}_{(a)b}$ (and the transformations they generate) is captured in the relationship $(\partial/\partial\theta)(\partial/\partial\bar\theta) + (\partial/\partial\bar\theta)(\partial/\partial\theta) = 0$. Thus, our present extension of earlier works [23–27] is a very natural generalization of the horizontality condition where the beauty of the geometrical interpretations is not spoiled in any way.

We dwell a bit on the negative sign in (4.6) for the expansion of the chiral superfield $\mathcal{B}(\tau, \theta)$. In fact, the bosonic ghost term $\dot{\bar\beta}\dot\beta$ in (2.7) (or (2.14)) remains invariant under $\beta \to \pm\bar\beta, \bar\beta \to \pm\beta$. We have taken the (+) sign for our description of the anti-BRST transformations $s^{(2)}_{ab}$. However, one could choose $\beta \to -\bar\beta, \bar\beta \to -\beta$ equally well. In that case, the negative sign in the expansion of (4.6) disappears. However, under the latter choice, the beautiful expansions of the superfields $E(\tau, \theta, \bar\theta)$ and $X_\mu(\tau, \theta, \bar\theta)$ in (4.16) get disturbed and some minus signs crop up in the expansion. This is why we have opted for the (+) sign in ($\beta \to \pm\bar\beta, \bar\beta \to \pm\beta$) and all the transformations, listed in the whole body of the present text, are consistent with it. Geometrically, it seems that the translation of the antichiral superfield $\bar{\mathcal{B}}(\tau, \bar\theta)$ along the $(+\bar\theta)$-direction of the supermanifold produces the BRST $s^{(2)}_b$ transformation for the anti-ghost field $\bar\beta$. However, the anti-BRST transformation $s^{(2)}_{ab}$ for the ghost field β is produced by the translation of the chiral superfield $\mathcal{B}(\tau, \theta)$ along the $(-\theta)$-direction of the supermanifold. This kind of discrepancy appears, perhaps, because of the peculiar behaviour of these bosonic (anti-) ghost fields which are commutative in nature but are restricted to be nilpotent of order two (i.e. $\beta^2 = 0, \bar\beta^2 = 0$).

In our present investigation, we have not dwelt on the derivation of the beautiful nilpotent symmetries $s^{(0)}_{(a)b}$ (cf. (2.6)) in the framework of the augmented superfield formalism because we cannot add the bosonic and fermionic 1-form super connections $\tilde V$ and $\tilde{\mathcal{F}}$ (defined in (3.3) and (4.1)) together. However, we strongly believe that if
(i) the action with Lagrangian (2.7) is written in terms of the superfields, and
(ii) the BRST symmetry transformations (2.6) are expressed in terms of the superfield too, we shall be able to derive these beautiful nilpotent symmetry transformations (i.e. without any restrictions) in the framework of augmented superfield approach to BRST formalism. We would like to lay stress on the fact that the nilpotent symmetry transformations in (2.6) are beautiful because there are no restrictions (i.e. $\beta^2 \neq 0, \bar\beta^2 \neq 0$) and no other peculiarities (like its being a composite of two fermions, etc.) are associated with the bosonic (anti-) ghost fields $(\bar\beta)\beta$. Furthermore, our approach could be extended to be applied to some more complicated and interesting field theoretic supersymmetric systems so that the ideas proposed in our present endeavour could be put on a firmer footing. These are some of the issues that are under investigation and we shall report these results in our forthcoming publications [33].

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